From (1) and (3),

$$x^2 + y^2 = (a^2 + b)/2a \quad \text{and} \quad 2xy = (2a^2 - b)/2a.$$

From these equations

$$x + y = \pm \sqrt{(3a^2 - b)/2a} \qquad \text{and} \qquad x - y = \pm \sqrt{(3b - 2a)/2a}.$$

Whence

$$x = \frac{1}{4a}[\pm \sqrt{6a^3 - 2ab} \pm \sqrt{6ab - 2a^3}],$$

$$y = \frac{1}{4a}[\pm \sqrt{6a^3 - 2ab} \mp \sqrt{6ab - 2a^3}].$$

Solved similarly by G. W. Hartwell, Paul Capron, Bernard Kramer, W. C. Eells, Clifford N. Mills, J. L. Riley, G. R. Mirick, H. C. Feemster, C. E. Githens, A. A. Nauer, Elmer Schuyler, and the Proposer.

Editorial Note.

These solutions are all conventional and do not state clearly,

(1) Which x's go with which y's. The four solutions obtained are, as to signs of the radicals,

$$\begin{array}{c|c|c|c|c} x & ++ & +- & -+ & -- \\ y & +- & ++ & -- & -+ \end{array}$$

(2) Necessary and sufficient conditions for real roots.

They are

$$a > 0, \quad b > 0, \quad 9a^2 > 3b > a^2.$$

(3) The number of solutions is not that demanded by Bezout's theorem. There is no way of being sure that all the real solutions have been found except by producing the complete set of eight solutions and discarding those not acceptable. In this case the loci (1) and (2) have ordinary contact at each of the complex points at infinity in the directions whose slopes are ω and ω^2 (cube roots of unity). This accounts for the other four points and clinches the argument as to the real intersections.

GEOMETRY.

421. Proposed by R. P. BAKER, University of Iowa.

Assuming the details of the proof of the existence of a sphere inscribed in a tetrahedron as usual in the texts, give an intuitional proof that there are in general eight spheres each touching the four faces, but for the regular tetrahedron only five. How many special types are there?

Solution by J. W. Clawson, Collegeville, Pa.

Call the planes passing through the six edges and bisecting the dihedral angles internally and externally respectively a, a'; b, b'; c, c'; d, d'; e, e'; f, f'. Call the planes BCD, CDA, DAB, and ABC I, II, III, and IV respectively (fig. 1).

A point will be said to be on the positive side of a face if it is on the same side of that face as a point inside the tetrahedron; if not it will be said to be on the negative side.

Now every point in a is equidistant from III and IV with same signs; every point in a' is equidistant from III and IV with opposite signs. So every point in b is equid stant from II and IV with same signs; every point n b' equidistant from II, IV with opposite signs. Similarly c and c' with respect to II and III;

d and d' with respect to I and IV; e and e' with respect to I and II; f and f' with respect to I and III.

It is easily proved, then, that a, b, c, d, e, f meet at O, a point equidistant from all faces and having the same sign with respect to all, i. e., the center of the inscribed sphere. Similarly that a, b, c, d', e', f' meet at E_7 a point equidistant from all four faces, the distances from II, III, IV having the same sign and that from I the opposite sign, i. e., the center of an escribed sphere touching the face I on the opposite side to the inscribed sphere. Similarly a, d, f, b', c', e' determine

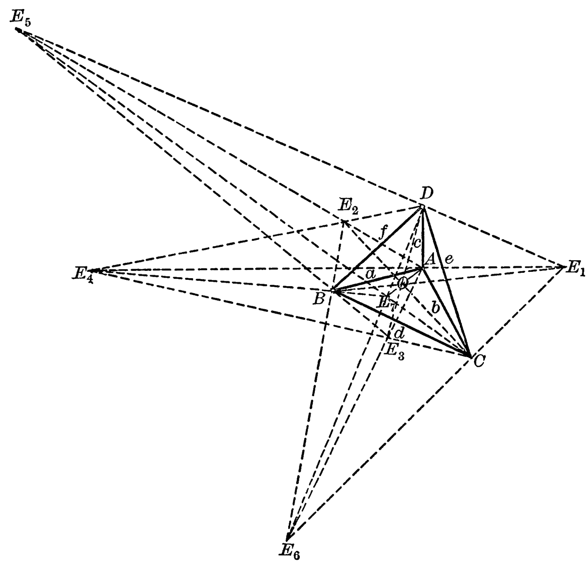

FIG. 1.

E_1, the center of the escribed sphere touching the face II opposite the inscribed sphere; b, d, e, a', c', f' determine E_2, the center of the escribed sphere touching face III opposite the inscribed sphere; c, e, f, a', b', d' determine E_3, the center of the escribed sphere touching the face IV opposite the inscribed sphere.

Again a, e, b', c', d', f' determine a point E_4, equidistant from all four faces, the distances from I and II having the same sign; and the distances from III and IV having the other sign. Similarly b, f, a', c', d', e' determine E_5, whose distances from I and III have the same sign and its distances from II and IV the other sign. And c, d, a', b', e', f' determine E_6, whose distances from I and IV have the same sign and its distances from II and III the other sign.

This gives us seven escribed spheres in all.

In the case where the tetrahedron is regular, three of these ex-centers, the three last named, E_4, E_5, E_6, are at infinity. To prove this, consider fig. 2. Bisect CD at F. The plane of ΔABF is perpendicular to edge CD and $\angle AFB$ is the plane angle of the dihedral angle CD. In ΔABF, if $AB = l$, $AF = BF = l\sqrt{3}/2$. Hence $\sin F/2 = 1/\sqrt{3} = \sin \angle AFI$. Draw planes bisecting AC,

CD, DA externally (planes b′, c', e'). These meet at E_1, forming a tetrahedron $ACDE_1$. Consider the vertex A. Face angle $CAD = 60°$, dihedral angle $AC = \cos^{-1} 1/\sqrt{3}$, dihedral angle $AD = \cos^{-1} 1/\sqrt{3}$. By spherical trigonometry, law of cosines, we find dihedral angle $AE_1 = 90°$. Similarly, considering vertices C and D, we find dihedral angles CE_1, DE_1 to be 90°. Hence $AE_1 \perp E_1C, E_1D$; $AE_1 \perp$ plane of CE_1D; $AE_1 \perp E_1F$. But IF is $\perp E_1F$, whence AE_1 is $\parallel$ to IF.

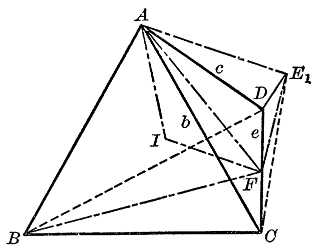

FIG 2.

Hence the line of intersection of a, b', and c' is parallel to the plane e and meets it only at infinity. So the point E_4 is at infinity. Similarly it may be shown that E_5 and E_6 are at infinity in this case.

This is the limiting case. In general AE_1 meets IF either on AE_1 produced or on E_1A produced.

Returning to fig. 1, A, E_1, E_4; A, E_2, E_5; A, E_3, E_6 are collinear. Also B, E_2, E_6; B, E_3, E_5; B, E_7, E_4; C, E_1, E_6; C, E_3, E_4; C, E_7, E_5; D, E_1, E_5; D, E_2, E_4; D, E_7, E_6.

Now if A, E_1, E_4 are collinear in that order (see fig. for Geom. 417), it is evident that the orders E_2, D, E_4; E_3, C, E_4; B, E_7, E_4; are determined. If A, E_2, E_5 are in that order, E_3, B, E_5; E_1, D, E_5; C, E_7, E_5 are determined orders. If A, E_3, E_6 are in that order, E_1, C, E_6; E_2, B, E_6; D, E_7, E_6 are determined orders.

If, however, E_1, A, E_4 be the order (see fig. 1), the orders D, E_2, E_4; C, E_3, E_4; E_7, B, E_4 obtain. And similarly with E_2, A, E_5 and E_3, A, E_6.

In special cases one, two, or three of the points E_4, E_5, E_6 may be at infinity. In that case the corresponding outer escribed sphere or spheres would not exist. We may therefore have four, five, six, or seven escribed spheres to a tetrahedron.

There are *two* types. (1) One of the vertices may be outside all three of the 'edgal' compartments in which the outer escribed spheres lie. In this case we may think of that vertex as at the top of the tetrahedron. The inscribed and three of the inner escribed spheres will then be above the level of the corresponding base, while the fourth inner and the three outer escribed spheres will be below that base. (2) One of the vertices may be at a corner of each of the three 'edgal' compartments in which the outer escribed spheres lie. In this case the three outer escribed spheres are above that vertex, the inscribed and three of the inner escribed spheres lie at a level between the vertex and its base and one of the inner escribed spheres only is below that base. It will be found

that if all seven escribed spheres are present the configuration can always be placed in one of these two types.

If one, two, or three of the outer spheres are lacking it will be found that we can place the configuration in *either* of the two types named above according as we consider the centers of the spheres which are at infinity to be *above* or *below* the vertex in question. These special cases occupy a limiting position between the two main types.

It may be pointed out that each vertex of the tetrahedron is collinear with four pairs of the eight points I, E_1, E_2, E_3, E_7, E_4, E_5, E_6. Fig. 1 shows the second type of tetrahedron; the figure that accompanies Geom. 417 shows the first type.

Editorial Note.—The discussion, scarcely intuitional, is printed for the division of the general case (7 finite ex-centers) into two sub-types. The intuitional view-point for the division into types with 4, 5, 6, 7 escribed spheres desired by the proposer is afforded by considering the angles at opposite pairs of edges. If equal the corresponding ex-center is at infinity, if unequal the ex-center is in the compartment of greater angle.

436. Proposed by A. J. KEMPNER, University of Illinois.

Given in a plane two similar curves arbitrarily situated, except that they shall possess the same sense of direction (which, of course, does not mean that they shall be similarly located). Let corresponding points on both curves be joined by straight lines, and let all of these straight lines be divided in the same ratio $\lambda : 1$, λ being any real number. Prove that the points of division all lie on a curve similar to the two given curves except when they all happen to fall together.

SOLUTION BY H. T. BIGELOW, Lafayette, Indiana.

Let the parametric equations of one of the curves be

$$x_1 = f(t), \qquad y_1 = \varphi(t). \tag{1}$$

The second curve, by reason of the similarity, is derivable from the first by an expansion from the origin, a rotation about the origin, and a translation. Its equation is, therefore,

$$\begin{aligned} x_2 &= a + k \cos \vartheta \cdot f(t) - k \sin \vartheta \cdot \varphi(t), \\ y_2 &= b + k \sin \vartheta \cdot f(t) + k \cos \vartheta \cdot \varphi(t), \end{aligned} \tag{2}$$

and corresponding points on the two curves (1) and (2) are given by equal values of t.

The corresponding point on the third curve is given by the equations

$$x_3 = \frac{x_1 + \lambda x_2}{1 + \lambda}, \qquad y_3 = \frac{y_1 + \lambda y_2}{1 + \lambda};$$

and

$$x_3 = \frac{\lambda a}{1 + \lambda} + \frac{(1 + \lambda k \cos \vartheta)}{1 + \lambda} f(t) - \frac{\lambda k \sin \vartheta}{1 + \lambda} \varphi(t),$$

$$y_3 = \frac{\lambda b}{1 + \lambda} + \frac{\lambda k \sin \vartheta}{1 + \lambda} f(t) + \frac{1 + \lambda k \cos \lambda}{1 + \lambda} \varphi(t).$$

If now we set $a' = \lambda a/(1 + \lambda)$ and $b' = \lambda b/(1 + \lambda)$ and choose k' and ϑ' so as to